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On Possibilities of Utilizing Various Conditions to Determine a Zero Set

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In an earlier paper by the authors in this issue (*J. Math. Anal. Appl.* **160**, 1991) various necessary conditions to determine the boundary of the zero set have been provided in a very general setting. Here, we classify and indicate which of the arsenal of conditions should be used for various classes of problems and functions for computational convenience.

I. INTRODUCTION

This paper complements our earlier paper [1] in this volume. The notation will be as in [1]. We show in [1] that various robust control problems can be reduced to the problem of locating certain zero sets. For

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the definition of the zero set and for the reduction mentioned above see the Introduction in [1].

As indicated in [1] the first stage in locating a zero set V in a given set G in $\mathbb{R}^m$ is the determination of the $(m-1)$-dimensional set L in $\mathbb{R}^m$ which contains the boundary of V relative to G. The main theorem and Theorems 1 and 2 of [1] give equations which represent necessary conditions for the boundary of V. By utilizing these equations one can get such a set L. There is a variety of ways of determining the set L from equations given by the main theorem, Theorem 1, and Theorem 2 of [1]. This flexibility and freedom in choosing the equaltions needed for determining such a set L contributes to a reduction of the computational complexity of this problem.

In the sequel the main theorem and Theorems 1 and 2 are referred to those of [1].

II. On the Various Possibilities of Combining the Main Theorem with Theorems 1 and 2

We will consider different values of d, r, and m (in the notation of the main theorem) and illustrate the application of Theorems 1 and 2 together with the main theorem in the determination of L.

A. *The Case* $d=1$, $r=2$, *and* $m=2$ (f is real and depends on ξ_1, ξ_2, s_1, and s_2). In this case the equations (2) and (3) of the main theorem are

$$f(\xi_1^0, \xi_2^0, s_1^0, s_2^0)=0 \tag{1}$$

$$f_{\xi_1}(\xi_1^0, \xi_2^0, s_1^0, s_2^0)=0 \tag{2}$$

$$f_{\xi_2}(\xi_1^0, \xi_2^0, s_1^0, s_2^0)=0. \tag{3}$$

Here $f_{\xi_i}=\partial f/\partial \xi_i$. Let $h=(f, f_{\xi_1}, f_{\xi_2})$ then

$$h'=\begin{bmatrix} 0 & 0 & f_{s_1} & f_{s_2} \\ f_{\xi_1\xi_1} & f_{\xi_1\xi_2} & f_{\xi_1 s_1} & f_{\xi_1 s_2} \\ f_{\xi_2\xi_1} & f_{\xi_2\xi_2} & f_{\xi_2 s_1} & f_{\xi_2 s_2} \end{bmatrix}=(h_{ij})_{i=1,2,3}^{j=1,2,3,4}.$$

There are four possible sub-cases according to the rank of h'. We discuss each of them separately and show how the main theorem and Theorems 1 and 2 can be implemented in the process of the determination of the set L.

(A-i) Rank $h'(A^0, s^0)=3$. In this case the number of independent equations in the main theorem is maximal and there are several ways of getting the desired 1-dimensional set in the (s_1, s_2)-plane:

(a) One can solve the system of equations (1)–(3) and obtain a representation of each variable as a function of one real parameter t, $0 < t < 1$. In particular,

$$\{(s_1(t), s_2(t)) : 0 < t < 1\} \tag{4}$$

is the desired 1-dimensional set in $\mathbb{R}^2$.

(b) One can solve either equations (1) and (2) or (1) and (3), for ξ_1 (or ξ_2) and s_1 (or s_2) in terms of the remaining variables. Suppose that we have

$$\xi_1 = \varphi_1(\xi_2, s_2)$$

and

$$s_1 = \varphi_2(\xi_2, s_2).$$

In this stage we denote

$$g(\xi_2, s_1, s_2) = s_1 - \varphi_2(\xi_2, s_2),$$

invoke Theorem 1, and obtain

$$g(\xi_2, s_1, s_2) = 0 \tag{5}$$

$$\frac{\partial g(\xi_2, s_1, s_2)}{\partial \xi_2} = 0. \tag{6}$$

At points where the second derivative of g with respect to ξ_2 does not vanish ξ_2 can be expressed as a function of s_2 from (6), and with (5), one gets a solution as in (4). At other points we consider (5), (6), and

$$\frac{\partial^2 g}{\partial \xi_2^2} = 0, \tag{7}$$

and proceed as before by checking the next derivative. One can continue along this line to higher derivatives.

A similar case is obtained by solving (1) for s_1 (or s_2) in terms of the remaining variables and letting g be defined accordingly.

(c) One can solve either equation (2) or (3), for ξ_1 (or ξ_2) in terms of the remaining variables. Suppose that we obtain

$$\xi_1 = \varphi(\xi_2, s_1, s_2).$$

In this stage we invoke Theorem 2 for either equations (1) and (2) or (1) and (3), and obtain

$$g(\xi_2, s_1, s_2) = f(\varphi(\xi_2, s_1, s_2), \xi_2, s_1 \cdot s_2) = 0 \tag{8}$$

$$\frac{\partial g(\xi_2, s_1, s_2)}{\partial \xi_2} = 0. \tag{9}$$

At points where the second derivative of g with respect to ξ_2 does not vanish ξ_2 can be expressed as a function of s_1 and s_2 from (9), and with (8), one gets a solution as in (4). At other points we consider (8), (9), and

$$\frac{\partial^2 g}{\partial \xi_2^2}=0, \tag{10}$$

and proceed as before by checking the next derivative. One can continue along this line to higher derivatives.

(A-ii) Rank $h'(A^0, s^0)=2$. In this case we have dependent equations in the main theorem and again there are several ways of getting the desired 1-dimensional set in the (s_1, s_2)-plane:

(d) If the first two or the first and the third rows in $h'(A^0, s^0)$ are linearly independent and not all the entries in

$$\Delta_1=\begin{vmatrix} h_{21} & h_{22} \\ h_{31} & h_{32} \end{vmatrix}$$

are equal to zero at the point (A^0, s^0), then one can proceed as in case (b).

(e) In case that the second and the third rows in $h'(A^0, s^0)$ are linearly independent and $\Delta_1 \neq 0$ at the point (A^0, s^0), then one can solve the equations (2)–(3) for ξ_1 and ξ_2 in terms of (s_1, s_2). The substitution of these solutions in (1) implies a dependence

$$\phi(s_1, s_2)=0, \tag{11}$$

which complies with our objective to obtain a 1-dimensional set of points (s_1, s_2) in $\mathbb{R}^2$ which satisfies (1)–(3).

(f) Let

$$\Delta_2=\begin{vmatrix} h_{21} & h_{23} \\ h_{31} & h_{33} \end{vmatrix}, \qquad \Delta_3=\begin{vmatrix} h_{21} & h_{24} \\ h_{31} & h_{34} \end{vmatrix},$$

$$\Delta_4=\begin{vmatrix} h_{22} & h_{23} \\ h_{32} & h_{33} \end{vmatrix}, \qquad \text{and} \qquad \Delta_5=\begin{vmatrix} h_{22} & h_{24} \\ h_{32} & h_{34} \end{vmatrix}.$$

If there is an index k in the set $\{2, 3, 4, 5\}$ such that $\Delta_k \neq 0$ at the point (A^0, s^0), then we proceed as in case (c).

(A-iii) Rank $h'(A^0, s^0)=1$ and rank $f'(A^0, s^0)=1$. In this case we again have dependent equations in the main theorem and again there are several ways of getting the desired 1-dimensional set in the (s_1, s_2)-plane:

(g) If there is a nonzero entry in Δ_1, at the point (A^0, s^0), say h_{21}, i.e., $f_{\xi_1\xi_1}(A^0, s^0) \neq 0$, then one can proceed as in case (b) or (c).

(h) All the entries of Δ_1 are equal to zero at (A^0, s^0), than we consider these conditions together with (1) and proceed as in case (b) or (c).

Remark 1. This consideration does not change for $m > 2$, except that the formulation and writing become more cumbersome. For $r > 2$ we have a similar but more complicated study since there are more possibilities.

B. *The Case* $d = 2$, $r = 2$, *and* $m = 2$ (f may be either a scalar complex-valued function or a two-dimensional real-valued function, and depends on ξ_1, ξ_2, s_1, and s_2). In this case the equations (2) and (3) of the main theorem are

$$f_1(\xi_1^0, \xi_2^0, s_1^0, s_2^0) = 0 \tag{12}$$

$$f_2(\xi_1^0, \xi_2^0, s_1^0, s_2^0) = 0 \tag{13}$$

$$J(\xi_1^0, \xi_2^0, s_1^0, s_2^0) = \frac{\partial(f_1, f_2)}{\partial(\xi_1, \xi_2)}(\xi_1^0, \xi_2^0, s_1^0, s_2^0) = 0. \tag{14}$$

Let $h = (f_1, f_2, J)$ then

$$h' = \begin{bmatrix} f_{1\xi_1} & f_{1\xi_2} & f_{1s_1} & f_{1s_2} \\ f_{2\xi_1} & f_{2\xi_2} & f_{2s_1} & f_{2s_2} \\ J_{\xi_1} & J_{\xi_2} & J_{s_1} & J_{s_2} \end{bmatrix} = (h_{ij})_{i=1,2,3}^{j=1,2,3,4}.$$

There are again four possible sub-cases according to the rank of h'. We discuss each of them separately and show how the main theorem and Theorems 1 and 2 can be implemented in the process of the determination of the set L.

(B-i) Rank $h'(A^0, s^0) = 3$. In this case the number of independent equations in the main theorem is maximal and there are several ways of getting the desired 1-dimensional set in the (s_1, s_2)-plane:

(i) One can solve the system of equations (12)–(14) and obtain a solution as in (4).

(j) One can solve the equations (12)–(13) for two variables in terms of the rest. Then one can choose and proceed along one of the following ways:

1. Solve for ξ_1 (or ξ_2) and s_1 (or s_2) in terms of the rest. In this case we proceed as in (b).

2. Solve for s_1 and s_2 in terms of ξ_1 and ξ_2. This means,

$$s_1 = \varphi_1(\xi_1, \xi_2)$$

and

$$s_2 = \varphi_2(\xi_1, \xi_2).$$

In this stage we denote

$$g(\xi_1, \xi_2, s_1, s_2) = (s_1 - \varphi_1(\xi_1, \xi_2), s_2 - \varphi_2(\xi_1, \xi_2)),$$

invoke Theorem 1, and obtain

$$g(\xi_1, \xi_2, s_1, s_2) = 0 \tag{15}$$

$$J(g) = \frac{\partial(\varphi_1, \varphi_2)}{\partial(\xi_1, \xi_2)} = 0. \tag{16}$$

At points where the first derivative of $J(g)$ with respect to ξ_1 (or ξ_2) does not vanish ξ_1 (or ξ_2) can be expressed as a function of ξ_2 (or ξ_1, respectively) from (16), and with (15), one gets a solution as in (4). At other points we consider (15), (16) and for instance

$$\frac{\partial J(g)}{\partial \xi_1} = 0, \tag{17}$$

and proceed as before by checking the next derivative. One can continue along this line to higher derivatives.

(k) If not all the entries of

$$\Delta_1 = \begin{vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{vmatrix}$$

are zero then we can solve the equation (12) or (13), for ξ_1 (or ξ_2) in terms of the remaining variables and proceed as in case (b) or (c) for the equations (12)–(13).

(l) Let

$$\Delta_2 = \begin{vmatrix} h_{11} & h_{12} \\ h_{31} & h_{32} \end{vmatrix} \quad \text{and} \quad \Delta_3 = \begin{vmatrix} h_{21} & h_{22} \\ h_{31} & h_{32} \end{vmatrix}.$$

If there is an index $k \in \{2, 3\}$ such that $\Delta_k \neq 0$ then one can solve (12), (14) or (13), (14) for ξ_1 and ξ_2 in terms of s_1 and s_2. Substituting this solution in (13), a solution as in (11) can be obtained.

(B-ii) Rank $f'(A^0, s^0) = 2$ and rank $h'(A^0, s^0) < 3$. In this case we have dependent equations in the main theorem and we can proceed as in cases (j) and (k).

A similar remark to Remark 1 is relevant here too, which pertains also to complex-valued functions ($d = 2$), as well as to the general case of vector-valued functions.

Reference

1. G. Fruchter, U. Srebro, and E. Zeheb, Conditions on the boundary of the zero set and application to stabilization of systems with uncertainty, *J. Math. Anal. Appl.* **160** (1991).